African Journal of Mathematics and Mathematical Sciences ISSN: 3821-435x Vol. 2 (2), pp. 038-046, February, 2014. Available online at www.internationalscholarsjournals.org © International Scholars Journals

Full Length Research Paper

Coupled fixed points results for W-Compatible Maps in symmetric G-Metric spaces

Sumitra Dalal[1] and Dimplekumar Chalishajar[2]

[1]College of Science, Jazan University, K.S.A. Saudi Arabia.
[2] Department of Applied Mathematics, Mallory hall, Virginia Military Institute, VA 24450, USA.
Corresponding author. E-mail: dipu17370@yahoo.com,Phone: 1-540-817-8105

Accepted 31 October, 2013

Mustafa,(2004) generalized the concept of metric spacesby introducing G-metric spacesand proved fixed point theorems for maps satisfying different contractive conditions[see Mustafa,(2004),(2006),(2008),(2009),(2010)]. In this article, we introduce the notion of w-compatible maps, b-coupled coincidence points and b-common coupled fixed points for non self maps and obtain fixed point results using these new notions in G-metric spaces. It is worth to mention that our results neither rely on completeness of the space nor the continuity of any mappings involved therein. Also, relevant examples have been cited to illustrate the effectiveness of our results. As an application,we demonstrate the existence of solution of system of nonlinear integral equations.

Key words: Coupled fixed point, w- compatible maps,G-metric spaces, Integral equations.

INTRODUCTION

Mustafa, (2004) introduced the concept ofG-metric spaces as a generalization of the metric spaces. In this type of space a non-negative real number is assigned to every triplet of elements. After the establishment of Banachcontraction principle by Mustafa et al.,(2008),several fixed point results have been proved in this space. Some of these works may be noted [see Mustafa,(2004),(2006),(2008),(2009),(2010)].

Coupled fixed point problems belong to a category of problems in fixed point theory in which much interest has been generated recently after the publication of a coupled contraction theorem by Bhaskar andLakshmikantham, (2006). One of the reasons for this interest is the application of these results for proving the existence and uniqueness of the solution of differentialequations, integral equations, Voltera integral and Fredholmintegral equations, and boundary value problems. For comprehensive description of such work, we refer to [Aydi, Samet and Vetro, (2011),Rao, Altun andBindu,(2011),Sumitra et.al,(2012-2013),Bhaskar andLakshmikantham, (2006) and Shatanawi, (2011)].

Jungck[(1976), (1986)] introduced the concept of commuting and compatible maps in metric spaces. Later on, these concepts were exploited by many authors like Fisher, (1983), Singh and Jain, (2005), Aage and Salunke, (2012), Jungck, (1976 and 1986), Aydi, Samet andVetro, (2011), andAbbas, Nazir andSaadati, (2011) etc. to prove a multitude of results of varied kind.In 2012, we established the notion of weakly compatible for coupled maps and proved coupled coincidence and coupled fixed point results and for our work we refer to Sumitra et.al,(2012-2013).

The intent of this paper is to introduce the concept of w-compatible maps, b-coupled coincidence points and b-common coupled fixed points for non self maps and obtain fixed point results using these new notions in symmetricG-metric spaces.

Preliminaries

Now, we give some preliminaries and basic definitions which are used throughout the paper. Mustafa et al, (2004)introduced the concept of G-metric spaces as follows:

Definition 2.1

Let X be a nonempty set and G: X × X × X $\rightarrow$ R^+be a function satisfying the following properties:
(G1) G(x, y, z) = 0 if x = y = z,
(G2) 0 < G(x, x, y),∀ x, y∈ X with x ≠ y,
(G3) G (x, x, y) ≤ G (x, y, z),∀ x, y, z ∈X with z ≠ y,
(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) =…… (symmetry in all three variables),
(G5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z),∀ x, y, z, a ε X, (rectangle inequality).
The function G is called a generalized metric or, more specifically, a G-metric on X andpair
(X, G) is called a G-metric space.
If condition (G6) is also satisfied then(X, G) is called symmetric G-metric space.
(G6) $G(x,x,y) = G(x,y,y),$ ∀ x, y ∈ X.
For more details on G-metric spaces, we refer the readers [Mustafa,(2004),(2006),(2008),(2009),(2010)].

Definition 2.2

Let (X, G) be a G-metric space. A point x∈ X is called a coincidence pointof self-mapsf and g on X if f(x) = g(x). In this case, w = f(x) = g(x) is called a point of coincidence of f and g.

Definition 2.3

Apair of self maps (f, g) of a G-metric space (X, G) is said to be weakly compatible if they commute at the coincidence points;
i.e., if f(u) = g(u) for some u in X, then (fg)(u) = (gf)(u). Here fg is the composition operator, i.e. f(g(u))=g(f(u)).

Definition 2.4.

An element (x, y) ∈X × X is called a coupled fixed point of the mapping
f: X × X → X if
f(x, y) = x, f(y, x) = y.

Definition 2.5
An element (x, y) ∈ X × X is called a coupled coincidence point of the mappings f: X × X → X and g: X → X if
f(x, y) = g(x), f(y, x) = g(y).

Definition 2.6

An element (x, y) ∈ X × X is called
(i) a common coupled fixed point of the maps f: X × X → X and g: X → X if
x = f(x, y) = g(x), y = f(y, x) = g(y).
(ii)a common fixed point of the maps f: X × X → X and g: X → X if
x = f(x, x) = g(x).

Definition 2.7.

The mapsf: X × X → X and g: X → X are said to be w-compatible if $g(f(x,y)) = f(g(x),g(y))$ whenever $f(x,y) = g(x)$ and $g(f(y,x)) = f(\mathrm{g(y)},\mathrm{g(x)})$ whenever $f(y,x) = g(y).$

Example 1. Let X = R andf: X × X → X and g: X → X be maps defined as
$f(x,y) = x^2 + y^2, g(x) = 2x$, then (1,1) and (0,0) are coupled coincidence point of f and g but the maps are not w-compatible as $gf(1,1) = g(2) = 4 \neq f(g(1),g(1)) = f(2,2) = 8.$

Example 2. LetX = R andf: X × X → X and g: X → X bemaps defined as
$f(x,y) = 3x + 2y - 6, g(x) = x$, then (1,2) and (2,1) are coupled coincidence point of f and g as $f(1,2) = 1 = g(1)$ and $f(2,1) = 2 = g(2).$

Also, $gf(x,y) = g(3x+2y-6) = 3x+2y-6 = f(g(x),g(y)) = f(x,y)$, which shows that f and g are w-compatible.
Now, we establish some new definitions for the development of our article:

Definition 2.8

An element (x, y) ∊ X × X is called a b-coupled coincidence point of the maps
F, g: X × X →X if
f(x, y) = g(x,y), f(y, x) = g(y,x).

Definition 2.9
An element (x, y) ∊ X × X is called a b-common coupled fixed point of the mapsf, g : X × X → X if
x= f(x, y) = g(x,y), y= f(y, x) = g(y,x).

Example 3. Let X = R and f, g : X × X → X be maps defined as
$f(x,y)=3x-2y+1, g(x,y)=2x-3y+2,$ then
$f(1,0)=4=g(1,0)$ and $f(0,1)=-1=g(0,1).$
Hence (1, 0) is a b-coupled coincidence point of f and g and (4,-1) is a b-coupled point of coincidence.
Example 4. Let X = R and f, g : X × X → X be maps defined as
$f(x,y)=\begin{Bmatrix} x+y-2, x<y \\ x-y+1, x\geq y \end{Bmatrix}, g(x,y)=\begin{Bmatrix} 2x-y+1, x<y \\ y-2x+5, x\geq y \end{Bmatrix}$, then
$f(1,2)=1=g(1,2)$ and $f(2,1)=2=g(2,1).$
Hence (1,2) is a b-common coupled fixed point of f and g.

Definition 2.10.

The mapsf,g : X × X → X are called w-compatible if
$f(g(x,y),g(y,x))=g(f(x,y),f(y,x))$ Wheneverf(x, y) = g(x,y) and f(y, x) = g(y,x).
Example 5. Let X = R andf, g : X × X → X be maps defined as
$f(x,y)=x+y, g(x,y)=x-y$, then (x,y) is a b-coupled coincidence point of f and g if and only if x=y. Moreover, we have $f(g(x,x),g(x,x))=g(f(x,x),f(x,x)), \forall x\in X$ and hence f and g are w-compatible.

MAIN RESULTS

Theorem 3.1.*Let (X, G) be a symmetric G-metric spaceandA, B:X× X →X be maps satisfying the following conditions:*
(3.1) A(X × X) ⊆ B(X × X).
(3.2) $\{B(x,y),B(y,x)\}$ *is a complete subspace of* $X\times X$, *∀ x,y ∈X.*
$$(3.3) G(A(x,y),A(u,v),A(u,v))\leq \alpha G\left(A(x,y),B(u,v),B(u,v)\right)+\beta G\left(B(x,y),A(u,v),A(u,v)\right)+$$
$$\gamma G\left(B(x,y),B(u,v),B(u,v)\right)+\delta G\left(A(u,v),B(x,y),B(x,y)\right),$$
$$\forall x,y,u,v\in X, \alpha,\beta,\gamma,\delta\geq 0 \text{ and } \alpha+2\beta+\gamma+2\delta<1.$$
Then A and B have a b-coupled coincidence point $(x,y)\in X\times X$ *i. e* $A(x,y)=B(x,y)$ *and* $A(y,x)=B(y,x).$
Moreover, if the pair (A, B) isw-compatible, then there exists unique x in X such that
A(x, x) = B(x, x).
Proof: Let x_0, y_0 be two arbitrary points in X. Since A(X × X) ⊆B(X × X), we can construct two sequences $\{z_n\}$ and $\{t_n\}$ in X such that
z_{2n}= A(x_{2n}, y_{2n}) = B(x_{2n+1}, y_{2n+1}) and t_{2n} = A (y_{2n}, x_{2n}) = B(y_{2n+1}, x_{2n+1}), for all
n≥ 0.
Step 1. We first show that $\{z_n\}$ and $\{t_n\}$ are Cauchy sequences. Using (3.3),
$$G(z_{2n},z_{2n+1},z_{2n+1})=G(A(x_{2n},y_{2n}),A(x_{2n+1},y_{2n+1}),A(x_{2n+1},y_{2n+1}))$$
$$\leq \alpha G\left(A(x_{2n},y_{2n}),B(x_{2n+1},y_{2n+1}),B(x_{2n+1},y_{2n+1})\right)$$

$+\beta G\left(B(x_{2n},y_{2n}),A(x_{2n+1},y_{2n+1}),A(x_{2n+1},y_{2n+1})\right)$
$+\gamma G\left(B(x_{2n},y_{2n}),B(x_{2n+1},y_{2n+1}),B(x_{2n+1},y_{2n+1})\right)$
$+\delta G\left(A(x_{2n+1},y_{2n+1}),B(x_{2n},y_{2n}),B(x_{2n},y_{2n})\right)$
$=\alpha G(z_{2n},z_{2n},z_{2n})+\beta G(z_{2n-1},z_{2n+1},z_{2n+1})$
$+\gamma G(z_{2n-1},z_{2n},z_{2n})+\delta G(z_{2n+1},z_{2n-1},z_{2n-1})$
$\leq 0+\beta G(z_{2n-1},z_{2n},z_{2n})+\beta G(z_{2n},z_{2n+1},z_{2n+1})$
$+\gamma G(z_{2n-1},z_{2n},z_{2n})+\delta G(z_{2n-1},z_{2n},z_{2n})+\delta G(z_{2n},z_{2n+1},z_{2n+1})$
i.e. $\left(1-\beta-\delta\right)G(z_{2n},z_{2n+1},z_{2n+1})\leq\left(\beta+\gamma+\delta\right)G(z_{2n-1},z_{2n},z_{2n})$

$$\Rightarrow G(z_{2n},z_{2n+1},z_{2n+1})\leq kG(z_{2n-1},z_{2n},z_{2n}),\ k=\frac{\left(\beta+\gamma+\delta\right)}{\left(1-\beta-\delta\right)}.$$

Similarly, we can have $G(z_{2n+1},z_{2n+2},z_{2n+2})\leq kG(z_{2n},z_{2n+1},z_{2n+1})$.
In general, $G(z_n,z_{n+1},z_{n+1})\leq kG(z_{n-1},z_n,z_n)\leq k^2G(z_{n-2},z_{n-1},z_{n-1}).....k^nG(z_0,z_1,z_1)$.
Therefore, for all $n,m\in N,\ n<m\ G(z_n,z_m,z_m)\leq G(z_n,z_{n+1},z_{n+1})+G(z_{n+1},z_{n+2},z_{n+2})+.....G(z_{m-1},z_m,z_m)$
$\leq\left(k^n+k^{n+1}+k^{n+2}.....k^{m-1}\right)G(z_0,z_1,z_1).$
Thus, $G(z_n,z_m,z_m)\to 0$ as $n,m\to\infty,$ similarly, we can show that $G(z_m,z_n,z_n)\to 0$ as $n,m\to\infty$ and hence $\{z_n\}$ is a Cauchy sequence in X . Similarly, we can show that $\{t_n\}$ is a Cauchy sequence.On theother hand, we have $\{z_n,t_n\}=\{B(x_n,y_n),B(y_n,x_n)\}\in\{B(x,y),B(y,x);x,y\in X\}$ is a complete subspace of $X\times X$, so $\exists(z,t)\in\{B(x,y),(y,x);x,y\in X\}$ such that $G(z_n,t_n)\to G(z,t).$ This implies that $\exists(x,y),(y,x)\in X\times X$ such that $z=B(x,y)$ and $t=B(y,x)$ with $z_n\to z=B(x,y)$ and $t_n\to t=B(y,x)$ as $n\to\infty.$ From (3.3), we have
$G\left(z_n,A(x,y),A(x,y)\right)=G\left(A(x_n,y_n),A(x,y),A(x,y)\right)$
$\alpha G\left(A(x_n,y_n),B(x,y),B(x,y)\right)+\beta G\left(B(x_n,y_n),A(x,y),A(x,y)\right)$
$+\gamma G\left(B(x_n,y_n),B(x,y),B(x,y)\right)+\delta G\left(A(x,y),B(x_n,y_n),B(x_n,y_n)\right).$
As $n\to\infty$,
$G\left(z,A(x,y),A(x,y)\right)\leq\alpha G\left(z,B(x,y),B(x,y)\right)+\beta G\left(z,A(x,y),A(x,y)\right)$
$+\gamma G\left(z,B(x,y),B(x,y)\right)+\delta G\left(A(x,y),z,z\right).$
$\Rightarrow G\left(B(x,y),A(x,y),A(x,y)\right)\leq\alpha G\left(B(x,y),B(x,y),B(x,y)\right)+\beta G\left(B(x,y),A(x,y),A(x,y)\right)$
$+\gamma G\left(B(x,y),B(x,y),B(x,y)\right)+\delta G\left(A(x,y),B(x,y),B(x,y)\right).$
$\Rightarrow\left(1-\beta\right)G\left(B(x,y),A(x,y),A(x,y)\right)\leq\delta G\left(A(x,y),B(x,y),B(x,y)\right).$

$$\Rightarrow G\left(B(x,y),A(x,y),A(x,y)\right)\leq\frac{\delta}{\left(1-\beta\right)}G\left(A(x,y),B(x,y),B(x,y)\right).$$

$\Rightarrow G\left(B(x,y),A(x,y),A(x,y)\right)\leq qG\left(A(x,y),B(x,y),B(x,y)\right)=qG\left(B(x,y),A(x,y),A(x,y)\right)$,
where $q=\dfrac{\delta}{\left(1-\beta\right)}$ and hence $A(x,y)=B(x,y).$ Similarly, we have $A(y,x)=B(y,x)$ and hence (x, y) is a coincidence point of the maps A and B.

Step 2. Let the pair (A, B) be w-compatible and $A(x,y)=B(x,y)$ and $A(y,x)=B(y,x),$ so

$A\left(B(x,y),B(y,x)\right)=B\left(A(x,y),A(y,x)\right)\Rightarrow A(z,t)=B(z,t)$ and
$A\left(B(y,x),B(x,y)\right)=B\left(A(y,x),A(x,y)\right)\Rightarrow A(t,z)=B(t,z).$

Using (3.3), we have

$$G\left(A\left(B(x_n,y_n),B(y_n,x_n)\right),A(x_n,y_n),A(x_n,y_n)\right)$$
$$\leq \alpha G\left(A\left(B(x_n,y_n),B(y_n,x_n)\right),B(x_n,y_n),B(x_n,y_n)\right)+\beta G\left(B\left(B(x_n,y_n),B(y_n,x_n)\right),A(x_n,y_n),A(x_n,y_n)\right)$$
$$+\gamma G\left(B\left(B(x_n,y_n),B(y_n,x_n)\right),B(x_n,y_n),B(x_n,y_n)\right)$$
$$+\delta G\left(A(x_n,y_n),B\left(B(x_n,y_n),B(y_n,x_n)\right),B\left(B(x_n,y_n),B(y_n,x_n)\right)\right).$$
$$G\left(A(z_n,t_n),z_n,z_n\right)\leq \alpha G\left(A(z_n,t_n),z_n,z_n\right)+\beta G\left(B(z_n,t_n),z_n,z_n\right)+\gamma G\left(B(z_n,t_n),z_n,z_n\right)$$
$$+\delta G\left(z_n,B(z_n,t_n),B(z_n,t_n)\right).$$

Taking $n\rightarrow\infty,\ G\left(A(z,t),z,z\right)\leq \alpha G\left(A(z,t),z,z\right)+\beta G\left(A(z,t),z,z\right)+\gamma G\left(A(z,t),z,z\right)$
$+\delta G\left(z,A(z,t),B(z,t)\right),$

which yields, $A(z,t)=z=B(z,t)$. Similarly, we have $A(t,z)=t=B(t,z).$

Step 3. Now,weclaim z = t. Again from (3.3), we have

$$G\left(A\left(x_n,y_n\right),A\left(y_n,x_n\right),A\left(y_n,x_n\right)\right)\leq \alpha G\left(A\left(x_n,y_n\right),B\left(y_n,x_n\right),B\left(y_n,x_n\right)\right)$$
$$+G\left(B\left(x_n,y_n\right),A\left(y_n,x_n\right),A\left(y_n,x_n\right)\right)$$
$$+\gamma G\left(B\left(x_n,y_n\right),B\left(y_n,x_n\right),B\left(y_n,x_n\right)\right)$$
$$+\delta G\left(A\left(y_n,x_n\right),B\left(x_n,y_n\right),B\left(x_n,y_n\right)\right).$$

i.e. $G\left(z_n,t_n,t_n\right)\leq(\alpha+\beta+\gamma+\delta)G\left(z_n,t_n,t_n\right).$

Taking $n\rightarrow\infty,$ we get z = t and hence the result follows.

Example 6. Let X = R and G be the G-metric on $X\times X\times X\rightarrow R$ defined as $G(x,y,z)=\left(|x-y|+|y-z|+|z-x|\right).$ Then (X, G) is a G-metric space**.** Define the mappings $A,B:X\times X\rightarrow X$ as follows:

$$A(x,y)=\begin{cases}\dfrac{x-y}{5}, x,y\in[0,1]\\ 1, otherwise\end{cases} \text{ and } B(x,y)=\begin{cases}x-y, x,y\in[0,2]\\ 1, otherwise\end{cases}.$$

Here, we see that $A\left(x,y\right)=\left[-\frac{1}{5},\frac{1}{5}\right]\subseteq B\left(x,y\right)=[-2,2]$ and $\left\{B\left(x,y\right),B\left(\mathrm{y,x}\right)\right\}=[-2,2]$ is complete subspace of $X\times X$, now from (3.3), we have

$$G(A(x,y),A(u,v),A(u,v))\leq\frac{1}{5}\left(2\left(|x-y|+|u-v|\right)\right)=\frac{1}{5}G(B(x,y),B(u,v),B(u,v))\leq G(B(x,y),B(u,v),B(u,v)),$$

proves contractive condition for $\alpha=\beta=\delta=0,\gamma=\frac{1}{5}$ and consequently, A and B have a b-coupled coincidence point. In this case, for any $x,y\in[0,1],(x,y)$ is a b-coupled coincidence pointif and only if x=y.

Moreover, we have $A(B(x,y),B(y,x))=A(0,0)=0=B(A(x,y),A(y,x))$.

Thus A and B are w-compatible, so by Theorem 3.1, we obtain the existence and uniqueness of b-common coupled fixed point of A and B and (0, 0) is the unique b-common coupled fixed point.

Theorem3.2. *Let (X, G) be a symmetric G-metric space and* $A: X\times X\to X$ *and* $S:X\to X$ *be maps satisfying the following conditions:*

(3.4) $A(X\times X)\subseteq S(X)$.

(3.5) $S(X)$ *is a complete subspace of* X.

$$(3.6)\ G(A(x,y),A(u,v),A(u,v))\le \alpha G\left(A(x,y),Su,Su\right)+\beta G\left(Sx,A(u,v),A(u,v)\right)+ \gamma G\left(Sx,Su,Su\right)+\delta G\left(A(u,v),Sx,Sx\right),$$

$\forall x,y,u,v\in X, \alpha,\beta,\gamma,\delta\ge 0$ *and* $\alpha+2\beta+\gamma+2\delta<1$. *Then A and S have a b-coupled coincidence point* $(x,y)\in X\times X$ *i.e* $A(x,y)=Sx$ *and* $A(y,x)=Sy$.

Moreover, if the pair (A, S) isw-compatible, then there exists unique x in X such that $A(x,x)=Sx$.

Proof: Consider the map $B: X\times X\to X$ defined by $B(x,y)=Sx, \forall x,y\in X$.

We will check that all the hypotheses of theorem 3.1are satisfied.

$(3.4)\Rightarrow(3.1)$ as $A(x,y)\subseteq Sx\Rightarrow A(x,y)\subseteq B(x,y), \forall x,y\in X$.

Also, the weak compatibility of the pair (A, S) yieldsthe w-compatibility of (A, B).The conditions(3.5) and (3.6) imply conditions (3.2) and(3.3) and so all the hypotheses of Theorem 3.1 are satisfied and hencethe maps A and S have a unique fixed point.

Theorem 3.3. *Let (X, G) be a symmetric G-metric spacesandP, S :* $X\to X$ *be the maps satisfying the following conditions:*

(3.7) $P(X)\subseteq S(X)$.

(3.8) $S(X)$ *is a complete subspace of* X.

$$(3.9)\ G(Px,Py,Py)\le \alpha G\left(Px,Sy,Sy\right)+\beta G\left(Sx,Py,Py\right)+\gamma G\left(Sx,Sy,Sy\right)+\delta G\left(Py,Sx,Sx\right)$$

$\forall x,y,u,v\in X, \alpha,\beta,\gamma,\delta\ge 0$ *and* $\alpha+2\beta+\gamma+2\delta<1$. *Then P and S have a b-coupled coincidence point* $(x,y)\in X\times X$ *i. e* $P(x,y)=Sx$ *and* $P(y,x)=Sy$.

Moreover, if the pair (P, S) isw-compatible, then there exists unique x in X such that $Px=Sx$.

Proof: Consider the mapsA, B: $X\times X\to X$ defined by

$A(x,y)=Px$ and $B(x,y)=Sx, \forall x,y\in X$.

We will check that all the hypotheses of Theorem 3.1 are satisfied.

$(3.7)\Rightarrow(3.1)$ as $Px\subseteq Sx\Rightarrow A(x,y)\subseteq B(x,y), \forall x,y\in X$.

Also, the w-compatibility of the pair (P, S) yieldsthe w-compatibility of (A, S). The conditions (3.8) and (3.9) imply conditions (3.2) and (3.3) and so all the hypotheses of Theorem 3.1are satisfied and hence the maps P and S have a unique fixed point.

APPLICATION

In this section, we study the existence of solutions of a system of nonlinear integral equations using the results proved in section 3.

Consider the following system of integral equations:

$$\left.\begin{aligned} P\left(x(t),\mathrm{y}(t)\right)&=\int_0^1 K_1(t,s,x(s),\mathrm{y}(s))ds+g(t),\\ P\left(\mathrm{y}(t),\mathrm{x}(t)\right)&=\int_0^1 K_1(t,s,\mathrm{y}(s),\mathrm{x}(s))ds+g(t),\end{aligned}\right\} \text{where } t\in[0,1]. \qquad (1)$$

Let $X=C\left(\left[0,1\right],R\right)$, be the set of continuous functions defined on $\left[0,1\right]$ endowed with G-metric

$$G(x,y,z)=\sup_{t\in[0,1]}\left|x(t)-y(t)\right|+\sup_{t\in[0,1]}\left|y(t)-z(t)\right|+\sup_{t\in[0,1]}\left|z(t)-x(t)\right|.$$

(Here, x, y, z are continuous functions).

Then (X, G) is a G-metric space.
Consider the following hypotheses hold:

(4.1) $K_1, K_2 : [0,1]\times[0,1]\times R \to R$ and $g : R \to R$ are continuous functions.

(i) For each $t, s \in [0,1]$

$$K_1\left(t, s, x(s), \mathrm{y}(s)\right) = K_2\left(t, s, \int_0^1 K_1\left(s, \tau, x(\tau), \mathrm{y}(\tau)\right) d\tau + g(s)\right),$$

$$K_2\left(t, s, x(s), \mathrm{y}(s)\right) = K_1\left(t, s, \int_0^1 K_2(s, \tau, x(\tau), \mathrm{y}(\tau) d\tau + g(s)\right).$$

(ii) $$\left|K_1\left(t, s, x(t), \mathrm{y}(t)\right) - K_1\left(t, s, \mathrm{u}(t), \mathrm{v}(t)\right)\right| \le \frac{1}{\kappa}\left|K_2\left(t, s, \mathrm{x}(t), \mathrm{y}(t)\right) - K_2\left(t, s, \mathrm{u}(t), \mathrm{v}(t)\right)\right|, \kappa \ge 1.$$

(4.2) $S : C\left([0,1], R\right)\times C\left([0,1], R\right) \to C\left([0,1], R\right)$ is a mapping satisfying

(i) For all $x, y \in C\left([0,1], R\right)$, there exist $\mathrm{u}, \mathrm{v} \in C\left([0,1], R\right)$ such that

$$S\left(x(t), \mathrm{y}(t)\right) = \int_0^1 K_2(t, s, x(s), \mathrm{y}(s)) ds + g(t),$$

$$S\left(\mathrm{y}(t), \mathrm{x}(t)\right) = \int_0^1 K_2(t, s, \mathrm{y}(s), \mathrm{x}(s)) ds + g(t), \text{ for all } t \in [0,1].$$

(ii) The set $\left\{S\left(x, y\right), S\left(\mathrm{y}, \mathrm{x}\right), x, y \in C\left([0,1], \mathrm{R}\right)\right\}$ is closed.

Remark: The above relations between $K_1 and K_2$ is satisfied for all functions u∈ $C\left([0,1], \mathrm{R}\right)$.

Now, we can formulate our result.

Theorem 4.1. *Under hypotheses (4.1) and (4.2), system of non-linear integral equations (1) has at least one solution in* $C\left([0,1], \mathrm{R}\right)$.

Proof: Define $\mathrm{P} : X \times X \to X$ by

$$P\left(x(t), \mathrm{y}(t)\right) = \int_0^1 K_1(t, s, x(s), \mathrm{y}(s)) ds + g(t),$$

$$P\left(\mathrm{y}(t), \mathrm{x}(t)\right) = \int_0^1 K_1(t, s, \mathrm{y}(s), \mathrm{x}(s)) ds + g(t).$$

It is obvious that (x, y) is a solution of (1) if and only if (x, y) is a b-coupled fixed point of maps P and S. To establish the existence of such point, we will use Theorem 3.1. For this let us check all the hypotheses of Theorem 3.1.

Hypotheses (3.1) and (3.2) follow directly from assumption (4.2). For (3.3), we see that for all $x, y, u, v \in X$, $\mathrm{t} \in [0,1]$, we have

$$G\left(P\left(x(t), \mathrm{y}(t)\right), P\left(\mathrm{u}(t), \mathrm{v}(t)\right), P\left(\mathrm{u}(t), \mathrm{v}(t)\right)\right)$$

$$= 2\sup_{t\in[0,1]}\left|P\left(\left(x(t), \mathrm{y}(t)\right)\right) - P\left(\mathrm{u}(t), \mathrm{v}(t)\right)\right|$$

$$= 2\sup_{t\in[0,1]}\left|\int_0^1 \left(K_1(t,s,x(s),\mathrm{y}(s) - K_1(t,s,\mathrm{u}(s),\mathrm{v}(s)ds\right)\right|$$

$$\leq \frac{2}{\kappa}\sup_{t\in[0,1]}\int_a^b \left|\left(K_2(t,s,x(s),\mathrm{y}(s) - K_2(t,s,\mathrm{u}(s),\mathrm{v}(s)\right)\right|ds$$

$$= \frac{1}{\kappa}G(S\left(x(t),\mathrm{y}(t)\right), S\left(\mathrm{u}(t),\mathrm{v}(t)\right), S\left(\mathrm{u}(t),\mathrm{v}(t)\right)), \kappa \geq 1.$$

$\forall$ x, y ∊ X and the condition (3.3) of Theorem 3.1 is satisfied for $\alpha = \beta = \delta = 0, \gamma = \frac{1}{\kappa}$.

Also,

$$P\left(x(t),\mathrm{y}(t)\right) = \int_0^1 K_1(t,s,x(s),\mathrm{y}(s))ds + g(t)$$

$$= \int_0^1 K_2\left(t,s,\int_0^1 K_1(s,\tau,x(\tau),\mathrm{y}(\tau))d\tau + g(s)\right)ds + g(t)$$

$$= \int_0^1 K_2\left(t,s,P\left(x(s),\mathrm{y}(s)\right)\right)ds + g(t) = SP\left(x(t),\mathrm{y}(t)\right).$$

$$S\left(x(t),\mathrm{y}(t)\right) = \int_0^1 K_2(t,s,x(s),\mathrm{y}(s))ds + g(t)$$

$$= \int_0^1 K_1\left(t,s,\int_0^1 K_2(s,\tau,x(\tau),\mathrm{y}(\tau))d\tau + g(s)\right)ds + g(t)$$

$$= \int_0^T K_1\left(t,s,S\left(x(s),\mathrm{y}(s)\right)\right)ds + g(t) = PS\left(x(t),\mathrm{y}(t)\right).$$

Thus, $PS\left(x(t),\mathrm{y}(t)\right) = SP\left(x(t),\mathrm{y}(t)\right)$ whenever $P\left(x(t),\mathrm{y}(t)\right) = S\left(x(t),\mathrm{y}(t)\right)$. Therefore, the pair (P, S) is w-compatible. Thus all the conditions of our Theorem 3.1 are satisfied and hence the conclusion follows.

Remark

Our work sets analogues, unifies, generalizes, extends and improves several well known results existing in literature, in particular the recent results of [1-5, 8-10, 20-21] etc. in the frame work of G-metric spaces as the notion of w-compatible is more general than commuting, weakly commuting and compatible maps. Our Theorems 3.1, 3.2 and 3.3 have been proved by assuming much weaker condition than in analogous results. The results concerning commuting, weakly commuting and compatible maps being extendable in the spirit of our theorems, can be extended verbatim by simply using wider class of w-compatibly in place of commuting, weakly commuting and compatibility maps. Moreover, our results don't need the maps to be continuous.For example, our Theorem3.3 generalizes and extends the following theorem.

Theorem3.1[10]. *Let (X, G) be a G-metric space. Let* $T, g : X \to X$ *be two maps such that*

$$\mathrm{G}\left(Tx,Ty,Tz\right) \leq k\,\mathrm{G}\left(gx,\mathrm{g}\,y,\mathrm{g}\,z\right),$$

for all x,y,z. Assume that T and g satisfy the following conditions:

(A1) $Tx \subseteq gx$.

(A2) gx *is G-complete.*

(A3) g is G-continuous and commutes with T.

if $k \in [0,1)$, *then there is a unique* $x \in X$ *such that* $gx = Tx = x$.

ACKNOWLEDGMENTS

The authors are grateful to both the learned referees for their fruitful suggestions and remarks towards the improvement of this paper. Second author is thankful to Virginia Military Institute (VMI), USA for providing research facility.

REFERENCES

Aage CT, Salunke JN (2012). Fixed Points For Weak Contractions In G-Metric Spaces, Applied Mathematics E-Notes, 12: pp. 23-28.

Abbas M, Nazir T, Saadati R (2011). Common fixed point results for three maps ingeneralized metric space , Abbas et al. Advances in Difference Equations, 2011: p. 49.

Agarwal RP, Karpinar E (2013). Remarks on some coupled fixed point theorems in G-metric spaces, Fixed Point Theory and Applications 2013.

Aydi H, Samet B, Vetro C (2011). Coupled fixed point results in cone metric spacesfor w-compatible mappings, Aydi et al. Fixed Point Theory and Applications p. 27.

Bhaskar TG, Lakshmikantham V (2006). Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis: Theory, Methods and Applications, 65(7): pp. 1379-1393.

Fisher B (1983). Common Fixed Point of Four Mappings, Bull. Inst. of Math.Academia.Sinica . 11: pp. 103-113.

Gugani M, Aggarwal M, Chugh R (2012). Common Fixed Point Results in G-Metric Spaces and Applications, Int. J. Comp. Appl. (0975 – 8887) April 2012. 43(11):pp 38-42

Jungck G (1976). Commuting mappings and fixed point, Amer. Math. Monthly, 83: pp. 261-263.

Jungck G (1986). Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9: pp. 771-779.

Mishra SN (1991). Common fixed points of Compatible mappings in PM-Spaces, Math.Japon 36: pp. 283-289.

Mustafa Z, Obiedat H (2010). A Fixed Point Theorem of Reich in G-Metric Spaces , CUBO A Math. J. 12(01): pp. 83–93. March 2010.

Mustafa Z, Obiedat H, Awawdeh F (2008). Some fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory and Applications, Volume 2008, Article ID 189870, doi:10.1155/2008/189870.

Mustafa Z, Shatanawi W, Bataineh M (2008). Fixed point theorem on uncomplete G-metric spaces, J. Math. Stat. 4: pp. ?196-2011.

Mustafa Z, Shatanawi W, Bataineh M (2009). Existence of fixed point results in G-metric spaces, Int. J. Math. Math.Sci. Volume 2009, Article ID 283028, doi:10.1155/2009/283028.

Mustafa Z, Sims B (2004). Some remarks concerning D-metric spaces, Proceedings of International Conference on Fixed Point Theory and Applications, pp. 189-198, Yokohama, Yokohama, Japan, 2004.

Mustafa Z, Sims B (2006). A new approach to generalized metric spaces, Journal of Nonlinear Convex Analysis, 7: pp. 289-297.

Mustafa Z, Sims B (2009). Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Applications, Volume 2009, Article ID 917175, doi: 10.1155 /2009 /917175.

Rao KPR, Altun I, Bindu SH (2011). Common Coupled Fixed-Point Theorems in Generalized FuzzyMetric Spaces, Hindawi Publishing Corporation, Advances in Fuzzy Systems. Volume 2011 Article ID 986748, 6 pagesdoi:10.1155/2011/986748.

Shatanawi V (2011). Some Fixed Point Theorems in Ordered G-metric Spaces and Applications, Hindawi Publishing Corporation Abstract and Applied Analysis, Volume 2011, Article ID 126205,11 pages doi 10.1155/2011/126205.

Shatanawi W (2011). Coupled Fixed Point Theorems in Generalized Metric Spaces, Hacettepe J. Math. Stat. 40(3): pp. 441 – 447.

Singh B, Jain S (2005). A fixed point theorem in Menger Space through weak compatibility, J. Math.Anal. Appl. 301: pp. 439-448.

Sumitra (2012). Coupled Fixed Point Theorem for Weakly Compatible Mappings in Intuitionistic Fuzzy Metric Spaces, Int. J. Comp. Appl. (0975 – 8887) 50(23):July 2012, pp28-32.

Sumitra (2013), Gregus type fixed point results for tangential mappings satisfying contractive condition in Fuzzy metric spaces, Advances in fixed point theory, 2(2): pp. 124-136, ISSN:2049-9337.

Sumitra (2013).Coupled Fixed Point Theorem for Compatible Maps and its Variants in Fuzzy Metric Spaces, Jorden J. Math.Statics. 6(2): pp. 141-155.

Sumitra (2013).CLRg property for Coupled Fixed Point Theorems in Fuzzy Metric Spaces, Fuzzy Sets and System – to appear

Sumitra Sunny, Chaun BD (2013). Pant, Coincidence and common fixed point theorems in intutionistic Fuzzy –metric Spaces, Far East J. Math. Sci. 79(1): pp. 25-48.

Sumitra, Alamgir Khan M (2012). CLRg Property for Coupled Fixed Point Theorems in Fuzzy Metric Spaces, Int. J. Appl. Physics Math. 2(5), September 2012.

Sumitra, Fatimah A (2012). Alshaikh,Coupled Fixed Point Theorems for a Pair of Weakly Compatible Maps Along with (CLRg) Property in Fuzzy Metric Spaces, Int. J. Appl. Physics Math. 2(5):September 2012, pp 345-347.